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ON CERTAIN FINAL MOTIONS IN THE n-BODY PROBLEM*

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Oscillatory motions and motions with capture are constructed in the n-body problem $(n > 3)$ of celestial mechanics, and it is shown that bounded, oscillatory and unbounded motions are possible as $t \to \infty$ and $t \to -\infty$ (t is the time) in any combinations.

Consider $n \geqslant 3$ material points (bodies) $P_0, P_1, \ldots, P_{n-1}$, interacting with each other according to Newton's law, with gravitational constant $\gamma > 0$.

Definition 1. We shall say that a capture has taken place if all polar distances between the bodies are bounded as $t \to \infty$ (t is the time), while when $t \to -\infty$, one of the polar distances will tend to infinity.

Definition 2. We shall call the motion oscillatory as $t \to \infty$ $(t \to -\infty)$, if the closure of the corresponding half-trajectory in configurational space is not compact as $t \to \infty$ $(t \to -\infty)$, but neither does it tend to infinity.

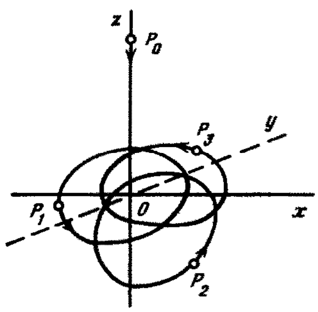

Fig.1

In the case of $n = 3$, the oscillatory motions and motions with capture were determined in /1-3/ for the special case of the three-body problem. The aim of the present paper is to extend the results to any $n > 3$. Let us consider the following special case of the n-body problem. We will assume that the bodies $P_1, \ldots, P_{n-1}$ of equal mass $m > 0$ move, in a certain rectangular system of coordinates x, y, z in such a manner that, in the plane $z = \text{const}$, they are always at the apices of a regular $(n-1)$-polygon whose centre lies on the z axis and the body P_0 of mass m_0 moves along the z axis (see the figure). We shall first assume that the mass $m_0 = 0$ and the bodies $P_1, \ldots, P_{n-1}$ are distributed in the plane $z = 0$. Then a motion of these bodies will exist, during which they will describe periodic trajectories, while remaining at all times at the apices of the regular $(n-1)$-polygon with centre at the origin of coordinates /4, p.109/.

Definition 3. We shall say that the motion of the body P_0, belonging to the system of n bodies $P_0, \ldots, P_{n-1}$, belongs to the class of symmetric models with parameters n, m (abbreviated to $SM(n, m)$) depending on the trajectories of the bodies $P_1, \ldots, P_{n-1}$, provided that the following conditions hold.

1). When $k \neq 0$, the mass of the body P_k in $m > 0$ and the mass of the body P_0 is zero.

2). The bodies $P_1, \ldots, P_{n-1}$ execute periodic motions in the plane $z = 0$, at the same time remaining at the apices of the regular $(n-1)$-polygon with centre at the point $O = (0, 0, 0)$ so that their trajectories never pass through the point O and the body P_0 moves along the z axis.

In Theorem 1 formulated below $x_k^{(n)}(t, m), y_k^{(n)}(t, m), z_k^{(n)}(t, m)$ are, respectively, the coordinates x, y, z, at the instant t, of the body P_k $(k = 0, 1, \ldots, n-1)$ for an arbitrary trajectory of the class $SM(n, m)$ and λ_n are given by

$$\lambda_n = \sum_{s=1}^{n-2} \left(\exp\left(2\pi i \frac{s}{n-1}\right) - 1\right) \left|\exp\left(2\pi i \frac{s}{n-1}\right) - 1\right|^{-3}$$

which by virtue of their definition are real and negative.

Theorem 1. For any $n \geqslant 3$ and any class $SM(3, m)$ there exists a class $SM(n, \lambda_3\lambda_n^{-1}m)$, such that the functions $z_0^{(3)} = z_0^{(3)}(t, m), z_0^{(n)} = z_0^{(n)}(t, \lambda_3\lambda_n^{-1}m)$ satisfy the equations

$$\partial^2 z_0^{(3)}/\partial t^2 = -Q_3(z_0^{(3)}, t), \quad \partial^2 z_0^{(n)}/\partial t^2 = -Q_n(z_0^{(n)}, t)$$

and the following relation holds for any z, t:

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$$Q_n(z,\ t) = {}^1/_2\,(n-1)\,Q_3(z,\ t)$$

Proof. Let us consider an arbitrary trajectory

$$(x_k^{(n)}, y_k^{(n)}, z_k^{(n)}) = (x_k^{(n)}(t), y_k^{(n)}(t), z_k^{(n)}(t)) \quad (k = 0, \ldots, n-1)$$

for which the motion of the body P_0 belongs to the class $SM\,(n, m)$. We introduce the complex variables $\xi_k = x_k^{(n)} + iy_k^{(n)}$ for $k = 1, \ldots, n-1$. Then the equation of motion of the body P_k $(k = 1, \ldots, n-1)$ can be written in the complex form

$$\frac{\partial^2 \xi_k}{\partial t^2} = m\gamma \sum_{l\neq k} (\xi_l - \xi_k)\, r_{kl}^{-3} \tag{1}$$

and equations of motion of the body P_0 can be written in the form

$$\frac{\partial^2 z_0^{(n)}}{\partial t^2} = -m\gamma \sum_{k=1}^{n-1} z_0^{(n)} r_{k0}^{-3} \tag{2}$$

where $r_{kl}\ (l = 0, \ldots, n-1)$ is the distance between the bodies P_k and P_l.

Let us now assume that $\xi_k = \zeta_k q_n (k = 1, \ldots, n-1)$, $\zeta_1, \ldots, \zeta_{n-1}$ are different complex numbers and $q_n = q_n(t, m)$ is a complex function different from zero. In this case Eqs.(1) and (2) become, respectively,

$$\zeta_k \frac{\partial^2 q_n}{\partial t^2} = q_n |q_n|^3 m\gamma \sum_{l\neq k} (\zeta_l - \zeta_k) |\zeta_l - \zeta_k|^{-3} \quad (k = 1, \ldots, n-1) \tag{3}$$

$$\frac{\partial^2 z_0^{(n)}}{\partial t^2} = -m\gamma z_0^{(n)} \sum_{k=1}^{n-1} (|q_n|^2 |\zeta_k|^2 + (z_0^{(n)})^2)^{-3/2} \tag{4}$$

According to (3) the expression

$$q^{1/2} \bar{q}_n^{\,3/2} \partial^2 q_n / \partial t^2 = c \tag{5}$$

where ($\bar{q}_n$ is a complex conjugate of q_n), is independent of t, and

$$\zeta_k c = m\gamma \sum_{l\neq k} (\zeta_l - \zeta_k) |\zeta_l - \zeta_k|^{-3} \quad (k = 1, \ldots, n-1) \tag{6}$$

Assuming now that the complex numbers ζ_k represent the apices of a right $(n-1)$-sided polygon inscribed in a circle of radius $r \neq 0$ with centre at the point O, we obtain the following relation for $k = 1, \ldots, n-1$:

$$\zeta_k^{-1} \sum_{l\neq k} (\zeta_l - \zeta_k) |\zeta_l - \zeta_k|^{-3} = \lambda_n r^{-3} \tag{7}$$

where λ_n is a number introduced in the notation. Therefore, by virtue of (6) and (7), Eq.(5) will take the form

$$q_n^{1/2} \bar{q}_n^{\,3/2} \partial^2 q_n / \partial t^2 = m\gamma\lambda_n r^{-3} \tag{8}$$

while Eq.(4) will become

$$\partial^2 z_0^{(n)} / \partial t^2 = -m\gamma z_0^{(n)} (n-1) (|q_n|^2 r^2 + (z_0^{(n)})^2)^{-3/2} = -Q_n(z_0^{(n)}, t) \tag{9}$$

If $n = 3$ and the masses of the bodies P_1 and P_2 are both equal to M, then the points P_1 and P_2 will move in the xy plane along periodic orbits, symmetrical about the origin of co-ordinates, and in this case the function $q_3 = q_3(t, M)$ will satisfy, according to (8) and (9), the equation

$$q_3^{1/2} \bar{q}_3^{\,3/2} \partial^2 q_3 / \partial t^2 = M\gamma\lambda_3 r^{-3} \tag{10}$$

and the function $z_0^{(3)} = z_0^{(3)}(t, M)$ will satisfy the equation

$$\partial^2 z_0^{(3)} / \partial t^2 = -2M\gamma z_0^{(3)} (|q_3|^2 r^2 + (z_0^{(3)})^2)^{-3/2} = -Q_3(z_0^{(3)}, t) \tag{11}$$

Replacing in Eq.(8) $q_n(t, m)$ by the function $q_3(t, M)$ we find, by virtue of (10), that Eq.(8) will hold if $m = M\lambda_3\lambda_n^{-1}$, and in this case we have $q_n(t, m) = q_3(t, M)$. Now equating Eqs. (9) and (11), we obtain the statement of Theorem 1.

Theorem 2. For any $n \geqslant 3$ there exists a class of $SM\,(n, m)$, in which capture and oscillatory motions take place, and bounded oscillatory and unbounded motions are possible in any combination as $t \to \infty$ and $t \to -\infty$.

Proof. Theorem 2 was proved in /2, 3/ for the case $n = 3$; these results were obtained as a corollary of analogous assertions concerning the solutions of the equation

$$\partial^2 z / \partial t^2 = -Q(z, t) \tag{12}$$

where $Q(z, t)$ is a smooth function 2π-periodic in t and satisfying certain general conditions (so that if the eccentricities of the orbits of the bodies P_1 and P_2 are small, the function $Q_3(z_0^{(3)}, t)$ in (11) will satisfy the same conditions as $Q(z, t)$ in (12)).

If on the other hand $n > 3$, then according to Theorem 1 a number $m > 0$ equal to the masses of the bodies $P_1, \ldots, P_{n-1}$ will exist such that the function $Q_n(z_0^{(n)}, t)$ in (9) will differ from the function $Q_3(z_0^{(3)}, t)$ by a constant factor $^1/_2(n-1)$. Therefore from /2, 3/ it follows that the function $Q_n(z_0^{(n)}, t)$ and Eq.(9) satisfy the same conditions as function $Q(z, t)$ and Eq.(12) in the case of a potential well of finite depth. The most difficult to confirm is the condition, according to which two closed curves corresponding to parabolic motions have a transversal point of intersection as $t \to \pm\infty$, (condition 8° in /3/, p.22). The check for the function $Q_n(z, t) = {}^1/_2(n-1)\,Q(z, t)$ is carried out in literally the same manner as the check in the proof of Theorem 9 of /3/ for the function $Q(z, t)$, since the constant factor $^1/_2(n-1)$ does not influence the proof itself.

Theorem 3. The assertion of Theorem 2 holds, if the mass m_0 of the body P_0 in the conditions of this theorem is sufficiently small and non-zero.

The proof of Theorem 3, taking Theorem 1 and 2 into account duplicates the proof of this theorem for the case $n = 3$ /5/.

A solution of the problem of capture is the n-body problem of celestial mechanics /6/ is a corollary of Theorems 2 and 3.

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